Examples of Einstein spacetimes with recurrent null vector fields

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Abstract

The Einstein Equation on 4-dimensional Lorentzian manifolds admitting recurrent null vector fields is discussed. Several examples of a special form are constructed. The holonomy algebras, Petrov types and the Lie algebras of Killing vector fields of the obtained metrics are found.

1 Introduction

We study the Einstein Equation on spacetimes admitting parallel distributions of null lines and construct several special examples.

Let (M, g) be a 4-dimensional Lorentzian manifold admitting a parallel distribution of null lines (i.e. (M, g) is *a Walker manifold*). This condition holds if and only if the holonomy group of (M, g) is contained in the subgroup $\mathrm{Sim}(2) \subset \mathrm{O}(1, 3)$ preserving a null line in the Minkowski space $\mathbb{R}^{1,3}$. Equivalently, (M, g) admits a recurrent null vector field in a neighborhood of each point, and locally there exist coordinates v, x^1, x^2, u (so called *Walker coordinates*) such that the metric g has the form

$$g = 2\mathrm{d}v\mathrm{d}u + h + 2A\mathrm{d}u + H \cdot (\mathrm{d}u)^2, \tag{1}$$

where $h = h_{ij}(x^1, x^2, u)\mathrm{d}x^i\mathrm{d}x^j$ is an u-dependent family of Riemannian metrics,
$A = A_i(x^1, x^2, u)\ \mathrm{d}x^i$ is an u-dependent family of one-forms, and H is a local function on M, [18]. The vector field $\partial_v = \frac{\partial}{\partial v}$ is null and recurrent, and it defines the parallel distribution of null lines. We will also write $x^1 = x$, $x^2 = y$.

A Lorentzian manifold (M, g) is called *an Einstein manifold* if g satisfies the equation

$$\mathrm{Ric} = \Lambda g, \qquad \Lambda \in \mathbb{R}, \tag{2}$$

where Ric is the Ricci tensor of the metric g, i.e. $\mathrm{Ric}_{ab} = R^c{}_{acb}$, where R is the curvature tensor of the metric g. The number $\Lambda \in \mathbb{R}$ is called *the cosmological constant.* If $\Lambda = 0$, i.e. $\mathrm{Ric} = 0$, than the manifold is called *Ricci-flat or vacuum Einstein.* We consider the case $\Lambda \neq 0$.

The Walker manifolds are of particular type of Kundt spaces [2, 4]. Recently G.W. Gibbons and C.N. Pope [8] considered the Einstein equation on Walker manifolds of arbitrary dimension.

A special example of the metric (1) is the metric of a pp-wave. It is given by $h = (dx)^2 + (dy)^2$, $A = 0$ and H independent of v. If such metric is Einstein, then it is vacuum Einstein, and

this happens whenever $(\partial_x^2 + \partial_y^2)H = 0$. For the plane waves ($H = A_{ij}(u)x^ix^j$), sometime it is useful to rewrite the metric in the Rosen coordinates, $g = 2dvdu + h$, where $h = C_{ij}(u)x^ix^j$ is a family of flat metrics, see e.g. [1, 15, 17]. The examples of the Einstein metrics that we construct have a similar structure.

Vacuum Einstein Walker metrics in dimension 4 are found in [12]. After a proper change of coordinates they are given by $h = (dx)^2 + (dy)^2$, $A_2 = 0$, $H = -(\partial_x A_1)v + H_0$, where A_1 is a harmonic function and H_0 can be found from a Poisson equation. In [5] several examples of such metrics are rewritten in new coordinates such that $A = 0$ and h is an u-family of flat metrics on $\mathbb{R}^2$.

In [13], all 4-dimensional Einstein Walker metrics with $\Lambda \neq 0$ are described. After a proper choice of the coordinates, h becomes an independent of u metric of constant curvature. Next, $A = Wdz + \bar{W}\bar{z}$, $W = i\partial_z L$, where $z = x + iy$, L is $\mathbb{R}$-valued function given by

$$L = 2\mathrm{Re}\left(\phi\partial_z(\ln P_0) - \frac{1}{2}\partial_z\phi\right), \quad 2P_0^2 = \left(1 + \frac{\Lambda}{|\Lambda|}z\bar{z}\right)^2, \tag{3}$$

where $\phi = \phi(z, u)$ is an arbitrary function holomorphic in z and smooth in u. Finally, $H = \Lambda^2 v + H_0$, where the function $H_0 = H_0(z, \bar{z}, u)$ can be found in a similar way.

An example of the Einstein Walker metric with $\Lambda \neq 0$ is given in [7]. After a change of the coordinates in [8], this metric is given by $A = 0$ and h independent of u. In [3], the universality of the metric from [7] and more generally of Einstein metrics with $\mathrm{Sim}(2)$-holonomy is proved.

The aim of this paper is to construct examples of Einstein Walker metrics with $\Lambda \neq 0$ such that $A = 0$ and h depends on u. The solutions from [13] are not useful for constructing examples of such form, since "simple" functions $\phi(z, u)$ define complicated forms A. First we find Walker metrics with h independent of u and $A \neq 0$, then we change the coordinates in such a way that $A = 0$. The constructed examples can be useful according to [2, 3, 4, 8]. Similar examples can be constructed in dimension 5, this dimension is discussed e.g. in [8, 9].

2 Coordinate transformations and reduction of the Einstein equation

The Einstein Equation (2) for the Walker metric (1) in arbitrary dimension is written down in [8]. It is shown that h is a family of Einstein Riemannian metrics with the cosmological constant Λ, $H = \Lambda v^2 + H_1 v + H_0$, $\partial_v H_1 = \partial_v H_0 = 0$, and there are three additional equations.

The Walker coordinates are not defined canonically and any other Walker coordinates $\tilde{v}, \tilde{x}^1, \tilde{x}^2, \tilde{u}$ such that $\partial_{\tilde{v}} = \partial_v$ are given by the transformation

$$\tilde{v} = v + f(x^1, x^2, u), \quad \tilde{x}^i = \tilde{x}^i(x^1, x^2, u), \quad \tilde{u} = u + c, \tag{4}$$

see [17, 5]. If the metric (1) is Einstein with $\Lambda \neq 0$, then there exist Walker coordinates such that $A = 0$ and $H_1 = 0$ [5]. Then the Einstein Equation takes the form

$$\Delta H_0 + \frac{1}{2}h^{ij}\ddot{h}_{ij} = 0, \quad \nabla^j \dot{h}_{ij} = 0, \quad h^{ij}\dot{h}_{ij} = 0, \quad \mathrm{Ric}_{ij} = \Lambda h_{ij}, \tag{5}$$

where $\dot{h}_{ij} = \partial_u h_{ij}$. Thus we get two equations on the family of Riemannian Einstein metrics and the Poisson equation for the function H_0. Of course, a solution gives the constant family,

$\dot{h}_{ij} = 0$ and a harmonic H_0, but it is more interesting to find solutions with $\dot{h}_{ij} \neq 0$. This will be done below.

In order to find the coordinates as in [5], we start with any Walker coordinates and consider two transformations. First consider the transformation $v \mapsto v + \frac{1}{2\Lambda}H_1$, after that $H_1 = 0$. Then consider the transformation $x^i \mapsto \tilde{x}^i(x^1, x^2, u)$ such that the inverse transformation satisfies the system of ordinary differential equations

$$\frac{\mathrm{d}x^i(u)}{du} = W^i(x^1(u), x^2(u), u),$$

where $W^i = -A_j h^{ij}$, with the initial conditions $x^i(u_0) = \tilde{x}^i$. The solution can be then written as $x^i = x^i(\tilde{x}^1, \tilde{x}^2, u)$.

Note that in dimension 2 (and 3) any Einstein Riemannian metric has constant sectional curvature, hence any such metrics with the same Λ are locally isometric, and the coordinates can be chosen in such a way that $\partial_u h = 0$. As in [13], it is not hard to show that if $\Lambda > 0$, then we may choose $h = (\mathrm{d}x)^2 + \sin^2 x\,(\mathrm{d}y)^2$, $H = \Lambda v^2 + H_0$, and the Einstein Equation is reduced to

$$\Delta_{S^2} f = -2f,\ \Delta_{S^2} H_0 = 2\Lambda\left(2f^2 - (\partial_x f)^2 + \frac{(\partial_y f)^2}{\sin^2 x}\right),\ \Delta_{S^2} = \partial_x^2 + \frac{\partial_y^2}{\sin^2 x} + \cot x\,\partial_x. \tag{6}$$

The function f defines the 1-form A: $A = -\frac{\partial_y f}{\sin x}\mathrm{d}x + \sin x\,\partial_x f \mathrm{d}y$. Similarly, if $\Lambda < 0$, then we chose $h = \frac{1}{-\Lambda \cdot x^2}\left((\mathrm{d}x)^2 + (\mathrm{d}y)^2\right)$, and get

$$\Delta_{L^2} f = 2f,\ \Delta_{L^2} H_0 = -4\Lambda f^2 - 2\Lambda x^2((\partial_x f)^2 + (\partial_y f)^2),\ \Delta_{L^2} = x^2(\partial_x^2 + \partial_y^2), \tag{7}$$

and $A = -\partial_y f \mathrm{d}x + \partial_x f \mathrm{d}y$. Thus in order to find partial solutions of (5) it is convenient to solve one of the above equations for f and then, changing the coordinates, to get rid of A. Note that *after such change h does not depend on u if and only if A is a Killing form for g* [8]. For $\Lambda > 0$ this happens if and only if $f = c_1(u)\sin x \sin y + c_2(u)\sin x \cos y + c_3(u)\cos x$; for $\Lambda < 0$ this happens if and only if $f = c_1(u)\frac{1}{x} + c_2(u)\frac{y}{x} + c_3(u)\frac{x^2+y^2}{x}$. The functions $\phi(z, u) = c(u)$, $c(u)z$ and $c(u)z^2$ from (3) define a Killing form A, see [5], and A becomes complicated for other ϕ.

Let now g be a Walker Einstein metric (1) with $A = 0$, and let $\tilde{h}(u)$ be a family of Riemannian metrics such that $h(u_0) = \tilde{h}(u_0)$ for some $u_0 \in \mathbb{R}$. Then *the metric g is isometric to the metric $\tilde{g} = 2\mathrm{d}v\mathrm{d}u + \tilde{h} + \tilde{H}(du)^2$ with some $\tilde{H}$ if and only if $h = \tilde{h}$ for all u*. Indeed, if the metrics are isometric, then one can be taken to another using transformation (4). Since $h(u_0) = \tilde{h}(u_0)$, we may assume that $\tilde{x}^k(x^1, x^2, u_0) = x^k$. Note that $A_i = \frac{\partial \tilde{x}^j}{\partial x^i}\left(\tilde{A}_j + \tilde{h}_{jk}\frac{\partial \tilde{x}^k}{\partial u}\right)$, hence $\frac{\partial \tilde{x}^k}{\partial u} = 0$. We conclude that $h = \tilde{h}$. The converse statement is trivial.

3 Curvature, holonomy and Petrov type

Let g be an Einstein metric of the form (1) with $\Lambda \neq 0$, $A = 0$ and $H = \Lambda v^2 + H_0$, i.e. $H_1 = 0$. Consider the vector fields $p = \partial_v$, $q = \partial_u - \frac{1}{2}H\partial_v$ and the distribution E generated by ∂_x and ∂_y. We will use the identification $\Lambda^2\mathbb{R}^{1,3} \simeq \mathfrak{so}(1,3)$. In [5] it is shown that the curvature tensor R of the metric g is given by

$$R(p,q) = \Lambda p \wedge q, \quad R(X,Y) = \Lambda X \wedge Y, \quad R(X,q) = -p \wedge T(X), \quad R(p,X) = 0,$$

where X, Y are sections of E, and $T(X) = -R(X,q)q$ is a symmetric endomorphism of E. It holds $T_i^j = -R_{4i4}^j$. Obviously, the metric g is indecomposable if and only if $T \neq 0$. In this case the holonomy algebra at any point $m \in M$ equals to $\mathbb{R}p_m \wedge q_m + \mathfrak{so}(E_m) + p_m \wedge E_m$ and it coincides with the maximal subalgebra $\mathfrak{sim}(2) \subset \mathfrak{so}(1,3)$ preserving the null line $\mathbb{R}p_m$. If the T is identically zero, then the manifold is decomposable and the holonomy algebra coincides with $\mathbb{R}p_m \wedge q_m \oplus \mathfrak{so}(E_m)$. The holonomy algebras of the Einstein Walker metrics are found also in each of the papers [16, 10, 6]. Remark that the manifolds under the consideration never admit (even locally) any null parallel vector field.

For the Weyl tensor we get

$$W(p,q) = \frac{\Lambda}{3} p\wedge q,\ W(p,X) = -\frac{2\Lambda}{3} p\wedge X,\ W(X,Y) = \frac{\Lambda}{3} X\wedge Y,\ W(X,q) = -\frac{2\Lambda}{3} X\wedge q - p\wedge T(X).$$

In [10, 11] it is shown that the Petrov type of the metric g is either II or D (and it may change from point to point), in particular, the manifolds under the consideration are *algebraically special* in the sense of the Petrov classification [14]. Using the Bel criteria and the above expression for W, it is easy to see that g has type II at a point $m \in M$ if and only if $T_m \neq 0$, and g has type D at a point $m \in M$ if and only if $T_m = 0$. Since T_m is symmetric and trace-free, it is either zero or it has rank 2. Hence, $T_m = 0$ if and only if $\det T_m = 0$.

4 Examples

Case $\Lambda < 0$. We consider a partial solution of Equation (7). This gives an Einstein metric. Then we find new coordinates such that $A = 0$ and h may depend on u.

Example 1 *Let $\partial_y f = 0$, then f satisfies $x^2 \partial_x^2 f - 2f = 0$. Hence, $f = \frac{c_1(u)}{x} + x^2 c_2(u)$. The function $\frac{c_1(u)}{x}$ defines a Killing form, and we take $f = c(u)x^2$, then $A = 2xc(u)\mathrm{d}y$, and we chose $H_0 = -\Lambda x^4 c^2(u)$. To get rid of A, we need to solve the system of equations*

$$\frac{dx(u)}{du} = 0, \qquad \frac{dy(u)}{du} = 2\Lambda c(u) x^3(u).$$

Imposing the initial conditions $x(0) = \tilde{x}$ and $y(0) = \tilde{y}$, we get that the inverse transformation to the required one has the form

$$v = \tilde{v}, \quad x = \tilde{x}, \quad y = \tilde{y} + 2\Lambda b(u)\tilde{x}^3, \quad u = \tilde{u},$$

where $b(u)$ is the function such that $\frac{\mathrm{d}b(u)}{\mathrm{d}u} = c(u)$ and $b(0) = 0$. With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}v\mathrm{d}u + h(u) + \left(\Lambda v^2 + 3\Lambda x^4 c^2(u)\right)(\mathrm{d}u)^2,$$

$$h(u) = \frac{1}{-\Lambda \cdot x^2}\Big(\left(36\Lambda^2 b^2(u) x^4 + 1\right)(\mathrm{d}x)^2 + 12\Lambda b(u) x^2 \mathrm{d}x\mathrm{d}y + (\mathrm{d}y)^2\Big).$$

Let $c(u) \equiv 1$, then $b(u) = u$. It holds $\det T = -9\Lambda^4 x^4 (x^4 + v^2)$. This shows that $\det T_m = 0$ ($m = (v,x,y,u)$) if and only if $v = 0$. In particular, the metric is indecomposable. The metric g is of Petrov type D on the set $\{(0,x,y,u)\}$ and it is of type II on its complement.

If $b(u) = u$, then the Lie algebra of Killing vector fields of the obtained metric is spanned by the vector fields ∂_y, $2v\partial_v + x\partial_x + y\partial_y - 2u\partial_u$, $\partial_u - 2\Lambda x^3 \partial_y$.

Example 2 *The functions* $f = x^2 y$ *and* $H_0 = -\Lambda x^4 y$ *are partial solutions of* (7). *Then,* $A = -x^2 \mathrm{d}x\mathrm{d}u + 2xy\mathrm{d}y$. *Consider the transformation with the inverse one given by*

$$v = \tilde{v},\ x = \tilde{x}(1 + 3\Lambda\tilde{u}\tilde{x}^3)^{-\frac{1}{3}},\ y = \tilde{y}(1 + 3\Lambda\tilde{u}\tilde{x}^3)^{\frac{2}{3}},\ u = \tilde{u}.$$

With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}v\mathrm{d}u + h(u) + \Lambda\left(v^2 + 3x^4y^2 + \frac{x^6}{\rho^2}\right)(\mathrm{d}u)^2, \qquad \rho = 1 + 3\Lambda u x^3,$$

$$h(u) = \frac{1}{-\Lambda}\left(\left(36\Lambda^2x^2y^2u^2 + \frac{1}{x^2\rho^2}\right)(\mathrm{d}x)^2 + 12\Lambda\rho y u \mathrm{d}x\mathrm{d}y + \frac{\rho^2}{x^2}(\mathrm{d}y)^2\right).$$

It can be checked that $\det T < 0$ *everywhere, hence the metric* g *is indecomposable and it is of Petrov type II everywhere.*

The Lie algebra of Killing vector fields of the obtained metric is spanned by the vector fields $3v\partial_v + x\partial_x + y\partial_y - 3u\partial_u$ *and* $\Lambda x^4\partial_x - 2\Lambda x^3 y\partial_y + \partial_u$.

The above two metrics are not isometric after any change of the functions H_0, see Section 2.

Case $\Lambda > 0$. Assuming $\partial_y f = 0$, we get the solution

$$f = c_1(u)\cos x\ + c_2(u)\left(\ln\left(\tan\frac{x}{2}\right)\cos x\ + 1\right)$$

of Equation (6). Obviously, yf is a solution of Equation (6) as well. The function $c_1(u)\cos x$ defines a Killing form A, and we omit it.

Example 3 *The function* $f = \ln\left(\tan\frac{x}{2}\right)\cos x\ + 1$ *is a solution of the first equation from* (6). *We get* $A = \left(\cos x\ - \ln\left(\cot\frac{x}{2}\right)\sin^2 x\right)\mathrm{d}y$. *Consider the transformation*

$$\tilde{v} = v, \quad \tilde{x} = x, \quad \tilde{y} = y - \Lambda u\left(\ln\left(\tan\frac{x}{2}\right) - \frac{\cos x}{\sin^2 x}\right), \quad \tilde{u} = u.$$

With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}v\mathrm{d}u + h(u) + \left(\Lambda v^2 + \tilde{H}_0\right)(\mathrm{d}u)^2,$$

$$h(u) = \left(\frac{1}{\Lambda} + \frac{4\Lambda u^2}{\sin^4 x}\right)(\mathrm{d}x)^2 + \frac{4u}{\sin x}\mathrm{d}x\mathrm{d}y + \frac{\sin^2 x}{\Lambda}(\mathrm{d}y)^2,$$

where $\tilde{H}_0$ *satisfies* $\Delta_h \tilde{H}_0 = -\frac{1}{2}h^{ij}\ddot{h}_{ij}$. *An example of such* $\tilde{H}_0$ *is*

$$\tilde{H}_0 = -\Lambda\left(\frac{1}{\sin^2 x} + \ln^2\left(\cot\frac{x}{2}\right)\right).$$

Coming back to the initial coordinates, we get $H_0 = \Lambda \cdot \left(\ln\left(\tan\frac{x}{2}\right)\cos x\ + 1\right)$. *We have*

$$\det T = -\frac{\Lambda^4}{\sin^4 x}\left(v^2 + \left(\ln\left(\cot\frac{x}{2}\right)\cos x\ - 1\right)^2\right).$$

Hence g *is of Petrov type D on the set* $\left\{(0, x, y, u) \middle| \ln\left(\cot\frac{x}{2}\right)\cos x\ - 1 = 0\right\}$ *and it is of type II on the complement to this set. The metric is indecomposable.*

The Lie algebra of Killing vector fields of the obtained metric is spanned by the vector fields ∂_y *and* $\partial_u + \Lambda\left(\frac{\cos x}{\sin^2 x} - \ln\left(\tan\frac{x}{2}\right)\right)\partial_y$.

Example 4 *The functions* $f = y\cos x$ *and* $H_0 = \Lambda\cdot\big(-y^2\cos^2 x + \ln\big(\tan\frac{x}{2}\big)\big)$ *are partial solutions of* (6). *We get* $A = -\cot x\mathrm{d}x - y\sin^2 x\,\mathrm{d}y$. *Consider the transformation*

$$\tilde{v} = v, \quad \tilde{x} = \arccos\left(e^{\Lambda u}\cos x\right), \quad \tilde{y} = ye^{-\Lambda u}, \quad \tilde{u} = u.$$

With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}v\mathrm{d}u + h(u) + \Lambda\left(v^2 - y^2e^{-2\Lambda u} + \frac{1}{2}\ln\frac{1-\rho}{1+\rho}\right)(\mathrm{d}u)^2, \quad \rho = e^{-\Lambda u}\cos x,$$

$$h(u) = \frac{e^{-2\Lambda u}\sin^2 x}{\Lambda\cdot(1-\rho^2)}(\mathrm{d}x)^2 + \frac{e^{2\Lambda u}(1-\rho^2)}{\Lambda}(\mathrm{d}y)^2.$$

It holds $\det T = -\frac{\Lambda^4\cdot\left(4y^2\cos^2 x + (\rho+2v)^2\right)}{4(1-\rho^2)^2}$. *Hence the metric is indecomposable and* T *vanishes at a point* (v,x,y,u) *if and only if* $y\cos x = 0$ *and* $e^{-\Lambda u}\cos x + 2v = 0$. *Consequently, the metric* g *is of Petrov type D at a point* (v,x,y,u) *if and only if* $x = \frac{\pi}{2}$ *and* $v = 0$, *or* $y = 0$ *and* $\cos x = -2ve^{\Lambda u}$. *It is of Petrov type II at other points.*

The Lie algebra of Killing vector fields of the obtained metric is spanned by the vector field $-\Lambda\cot x\partial_x - \Lambda y\partial_y + \partial_u$.

The metrics of the last two examples are not isometric after any change of H_0.

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